The inversion formula for automorphisms of the Weyl algebras and polynomial algebras

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Abstract

Let A_n be the n'th *Weyl algebra* and P_m be a polynomial algebra in m variables over a field K of characteristic zero. The following characterization of the algebras $\{A_n \otimes P_m\}$ is proved: *an algebra A admits a finite set $\delta_1, \ldots, \delta_s$ of commuting locally nilpotent derivations with generic kernels and $\cap_{i=1}^s \ker(\delta_i) = K$ iff $A \simeq A_n \otimes P_m$ for some n and m with $2n + m = s$, and vice versa.* The inversion formula for automorphisms of the algebra $A_n \otimes P_m$ (and for $\widehat{P}_m := K[[x_1, \ldots, x_m]]$) has found (giving a *new* inversion formula even for polynomials). Recall that (see [3]) *given $\sigma \in \mathrm{Aut}_K(P_m)$, then* $\deg \sigma^{-1} \leq (\deg \sigma)^{m-1}$ (the proof is *algebro-geometric*). We extend this result (using [non-holonomic] *$\mathcal{D}$-modules*): *given $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$, then $\deg \sigma^{-1} \leq (\deg \sigma)^{2n+m-1}$.* Any automorphism $\sigma \in \mathrm{Aut}_K(P_m)$ is determined by its *face* polynomials [8], a *similar* result is proved for $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$.

One can amalgamate two old open problems (**the Jacobian Conjecture** and **the Dixmier Problem**, see [6] problem 1) into a single question, (**JD**): *is a K-algebra endomorphism $\sigma : A_n \otimes P_m \to A_n \otimes P_m$ an algebra automorphism provided* $\det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^* := K \backslash \{0\}$? ($P_m = K[x_1, \ldots, x_m]$). It follows immediately from the inversion formula that *this question has an affirmative answer iff both conjectures have* (see below) [*iff one of the conjectures has a positive answer* (as it follows from the recent paper [5])].

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1   Introduction

The following notation will remain **fixed** throughout the paper (if it is not stated otherwise): K is a field of characteristic zero (not necessarily algebraically closed), module means a *left* module, $A_n = \oplus_{\alpha \in \mathbb{N}^{2n}} K x^\alpha$ is the n'th *Weyl algebra* over K, $P_m = \oplus_{\alpha \in \mathbb{N}^m} K x^\alpha$ is a *polynomial* algebra over K, $A := A_n \otimes P_m = \oplus_{\alpha \in \mathbb{N}^s} K x^\alpha$, $s := 2n + m$, is the Weyl algebra with polynomial coefficients where $x_1, \ldots, x_s$ are the canonical generators for A (see below). Any K-algebra automorphism $\sigma \in \mathrm{Aut}_K(A)$ is uniquely determined by the elements $x_i' := \sigma(x_i) = \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha x^\alpha$, $i = 1, \ldots, s$, $\lambda_\alpha \in K$, and so does its inverse, $\sigma^{-1}(x_i) = \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha' x^\alpha$, $i = 1, \ldots, s$.

What is <u>the</u> inversion formula for $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$**?** A natural (shortest) answer to this question is a formula for the coefficients $\lambda_\alpha' = \lambda_\alpha'(\lambda_\beta)$ *like the inversion formula (the*

Kramer's formula) in the linear polynomial case: given $x' = Ax$ where $A = (a_{ij}) \in \mathrm{GL}_m(K)$ (i.e. $x'_i = \sum_{j=1}^m a_{ij} x_j$ where $a_{ij} = \frac{\partial x'_i}{\partial x_j}$) then

$$ x = A^{-1} x' = (\frac{\partial x'_i}{\partial x_j})^{-1} x' = (\det A)^{-1} (\Delta_{ij}) x' $$

where Δ_{ij} are complementary minors for the matrix $(\frac{\partial x'_i}{\partial x_j})$, they are linear combinations of products of partial derivatives $\frac{\partial x'_i}{\partial x_j}$. So, the inversion formula in general situation is a formula, $\lambda'_\alpha = \lambda'_\alpha(\lambda_\beta)$, where *only* additions and multiplications are allowed of 'partial derivatives' of the elements x' (taking partial derivatives 'correspond' to operation of taking coefficients of x'). So, *the* inversion formula is the most *economical* formula (the point I want to make is that $x = \frac{1}{2}x'$ is the inversion formula for the equation $x' = 2x$ but $x = \frac{1}{2}(x' + 2 \int_0^1 f(t)dt + 2\dim_K \mathrm{Ext}_B^i(M,N)) - \int_0^1 f(t)dt - \dim_K \mathrm{Ext}_B^i(M,N)$ is 'not').

Theorem 2.4 gives *the inversion formula* for an automorphism $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. Theorem 4.3 gives a *similar formula* for an automorphism $\sigma \in \mathrm{Aut}_K(K[[x_1, \ldots, x_m]])$. For *another* inversion formula for $\sigma \in \mathrm{Aut}_K(P_m)$ see [3], [1].

The degree of σ^{-1} where $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. We extend the following result which according to the comment made on p. 292, [3]: 'was "well-known" to the classical geometers' and 'was communicated to us [H. Bass, E. H. Connell, D. Wright] by Ofer Gabber. ... He attributes it to an unrecalled colloquium lecture at Harward.'

Theorem 1.1 *[3], [9].* *Given* $\sigma \in \mathrm{Aut}_K(P_m)$, *then* $\deg \sigma^{-1} \leq (\deg \sigma)^{m-1}$.

The proof of this theorem is *algebro-geometric* (see [2] for a generalization of this result for certain varieties)). We extend this result (see Section 3).

Theorem 1.2 *Given* $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. *Then* $\deg \sigma^{-1} \leq (\deg \sigma)^{2n+m-1}$.

*Non-holonomic $\mathcal{D}$-modules are used in the proof (it looks like this is one of the first instances where *non-holonomic $\mathcal{D}$-modules* are of real use).

The algebras $\{A_n \otimes P_m\}$ as a class. Theorem 5.3 gives a characterization of the algebras $\{A_n \otimes P_m\}$ as a class via commuting sets of locally nilpotent derivations: *an algebra A admits a finite set $\delta_1, \ldots, \delta_s$ of commuting locally nilpotent derivations with generic kernels and $\cap_{i=1}^s \ker(\delta_i) = K$ iff $A \simeq A_n \otimes P_m$ for some n and m with $2n + m = s$, and vice versa* (the kernels $\ker(\delta_i)$ are *generic* if the intersections $\{\cap_{i \neq j} \ker(\delta_i) \,|\, j = 1, \ldots, s\}$ are *distinct*).

Left and right faces of an automorphism $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. Let $P_m = K[X_1, \ldots, X_m]$ be a polynomial algebra. For each $i = 1, \ldots, m$, the *algebra epimorphism* $f_i : P_m \to P_m/(X_i)$, $p \mapsto p + (X_i)$, is called the *face homomorphism*. J. McKay and S. S.-S. Wang [8] proved: *given $\sigma, \tau \in \mathrm{Aut}_K(P_m)$ such that $f_i \sigma = f_i \tau$, $i = 1, \ldots, m$, then $\sigma = \tau$*. So, an automorphism $\sigma \in \mathrm{Aut}_K(P_m)$ is completely determined by its *faces* $\{f_i \sigma \,|\, i = 1, \ldots, m\}$ or equivalently by its *face polynomials* $\{f_i \sigma(X_j) \,|\, i, j = 1, \ldots, m\}$ since each $f_i \sigma$ is an *algebra homomorphism*.

For the algebra $A := A_n \otimes P_m = K\langle x_1, \ldots, x_s \rangle$, $s := 2n + m$, (where $x_1, \ldots, x_s$ are the canonical generators) we have *left faces* $l_i : A \to A/Ax_i$, $a \mapsto a + Ax_i$, and *right faces* $r_i : A \to A/x_i A$, $a \mapsto a + x_i A$, $i = 1, \ldots, s$. These are homomorphisms of *left* and *right* A-modules rather then homomorphisms of algebras (if $x_i \in P_m$ then $l_i = r_i$ is an *algebra homomorphism*).

Theorem 6.1 states that: *given $\sigma, \tau \in \mathrm{Aut}_K(A)$ such that $r_i \sigma = r_i \tau$, $i = 1, \ldots, s$ then $\sigma = \tau$ (similarly, $l_i \sigma = l_i \tau$, $i = 1, \ldots, s$, imply $\sigma = \tau$).*

2 The Inversion Formula

In this section, the inversion formula (Theorem 2.4) is given.

Let A be an algebra over a field K and let δ be a K-derivation of the algebra A. For any elements $a, b \in A$ and a natural number n, an easy induction argument gives

$$\delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(a) \delta^{n-i}(b).$$

It follows that the kernel $A^\delta := \ker \delta$ of δ is a subalgebra (of *constants* for δ) of A and the union of the vector spaces $N := N(\delta, A) = \cup_{i \geq 0} N_i$ is a positively *filtered* algebra ($N_i N_j \subseteq N_{i+j}$ for all $i, j \geq 0$). Clearly, $N_0 = A^\delta$ and $N := \{a \in A \mid \delta^n(a) = 0 \text{ for some natural } n\}$.

A K-derivation δ of the algebra A is a *locally nilpotent* derivation if for each element $a \in A$ there exists a natural number n such that $\delta^n(a) = 0$. A K-derivation δ is locally nilpotent iff $A = N(\delta, A)$.

Given a ring R and its derivation d. The *Ore extension* $R[x; d]$ of R is a ring freely generated over R by x subject to the defining relations: $xr = rx + d(r)$ for all $r \in R$. $R[x; d] = \oplus_{i \geq 0} Rx^i = \oplus_{i \geq 0} x^i R$ is a left and right free R-module. Given $r \in R$, a derivation $(\mathrm{ad}\, r)(s) := [r, s] = rs - sr$ of R is called an *inner* derivation of R.

Lemma 2.1 *Let A be an algebra over a field K of characteristic zero and δ be a K-derivation of A such that $\delta(x) = 1$ for some $x \in A$. Then $N(\delta, A) = A^\delta[x; d]$ is the Ore extension with coefficients from the algebra A^δ, and the derivation d of the algebra A^δ is the restriction of the inner derivation $\mathrm{ad}\, x$ of the algebra A to its subalgebra A^δ. For each $n \geq 0$, $N_n = \oplus_{i=0}^{n} A^\delta x^i = \oplus_{i=0}^{n} x^i A^\delta$.*

Proof. For each element $c \in C := A^\delta$,

$$\delta([x, c]) = [\delta(x), c] + [x, \delta(c)] = [1, c] + [x, 0] = 0,$$

thus $d(C) \subseteq C$, and d is a K-derivation of the algebra C.

First, we show that the K-subalgebra N' of $N := N(\delta, A)$ generated by C and x is the Ore extension $C[x; d]$. We have $N' = \sum_{i \geq 0} Cx^i$ since, for each $c \in C$, $xc - cx = d(c)$. So,

it remains to prove that the sum $\sum_{i \geq 0} Cx^i$ of left C-modules is a direct sum. Suppose this is not the case, then there is a nontrivial relation of degree $n > 0$,

$$c_0 + c_1 x + \cdots + c_n x^n = 0, \quad c_i \in C, \quad c_n \neq 0.$$

We may assume that the degree n of the relation above is the *least* one. Then applying δ to the relation above we obtain the relation

$$c_1 + 2c_2 x + \cdots + nc_n x^{n-1} = 0$$

of smaller degree $n - 1$ since $nc_n \neq 0$ (char $K = 0$), a contradiction. So, $N' = C[x; d]$.

It remains to prove that $N = N'$. The inclusion $N' \subseteq N$ is obvious. In order to prove the inverse inclusion it suffices to show that all subspaces N_i belong to N'. We use induction on i. The base of the induction is trivial since $N_0 = C$. Suppose that $i > 0$, and $N_{i-1} \subseteq N'$. Let u be an arbitrary element of N_i. Then $\delta(u) \in N_{i-1} \subseteq N'$. For an arbitrary element $a = \sum c_j x^j \in N'$, we have $\delta(b) = a$ where $b = \sum (j+1)^{-1} c_j x^{j+1} \in N'$. Therefore, in the case of $a = \delta(u) \in N'$, we have $\delta(u) = \delta(b)$ for some $b \in N'$. Hence, $\delta(u - b) = 0$, and $u \in b + C \subseteq N'$. This means that $N = N'$, as required. $\square$

Theorem 2.2 *Let A be an algebra over a field K of characteristic zero, δ be a locally nilpotent K-derivation of the algebra A such that $\delta(x) = 1$ for some $x \in A$. Then the K-linear map $\phi := \sum_{i \geq 0} (-1)^i \frac{x^i}{i!} \delta^i : A \to A$ satisfies the following properties:*

1. *ϕ is a homomorphism of right A^δ-modules.*

2. *ϕ is a projection onto the algebra A^δ:*

$$\phi : A = A^\delta \oplus xA \to A^\delta \oplus xA, \quad a + xb \mapsto a, \quad \text{where} \quad a \in A^\delta, \ b \in A.$$

   In particular, $\operatorname{im}(\phi) = A^\delta$ and $\phi(y) = y$ for all $y \in A^\delta$.

3. *$\phi(x^i) = 0, \ i \geq 1$.*

4. *ϕ is an algebra homomorphism provided $x \in Z(A)$, the centre of the algebra A.*

Proof. The map ϕ is well-defined since δ is a locally nilpotent derivation. It is obvious that ϕ is a homomorphism of right A^δ-modules, $\phi(x) = 0$, and $\phi(y) = y$ for all $y \in A^\delta$. An easy computation shows that $\delta\phi(z) = 0$ for all $z \in A$, hence $\operatorname{im}(\phi) = A^\delta$.

If x is a *central* element then for $a, b \in A$:

$$
\begin{aligned}
\phi(ab) &= \sum_{i \geq 0} (-1)^i \frac{x^i}{i!} \delta^i(ab) = \sum_{i \geq 0} (-1)^i \frac{x^i}{i!} \sum_{j+k=i} \binom{i}{j} \delta^j(a) \delta^k(b) \\
&= \left(\sum_{j \geq 0} (-1)^j \frac{x^j}{j!} \delta^j(a) \right) \left(\sum_{k \geq 0} (-1)^k \frac{x^k}{k!} \delta^k(a) \right) = \phi(a)\phi(b).
\end{aligned}
$$

4

For a *not* necessarily central x, repeating the above computations we have, $\phi(x^i) = \phi(xx^{i-1}) = \phi(x)\phi(x^{i-1}) = 0$. Note that $A = \oplus_{i \geq 0} x^i A^\delta$ (Lemma 2.1). Then ϕ is a projection onto A^δ, $\phi : a_0 + xa_1 + \cdots \mapsto a_0$, where $a_i \in A^\delta$. $\square$

The *Weyl* algebra $A_n = A_n(K)$ is a K-algebra generated by $2n$ generators $q_1, \ldots, q_n$, $p_1, \ldots, p_n$ subject to the defining relations:

$$[p_i, q_j] = \delta_{ij}, \quad [p_i, p_j] = [q_i, q_j] = 0 \quad \text{for all} \quad i, j,$$

where δ_{ij} is the Kronecker delta, $[a, b] := ab - ba$.

For the algebra $A_n \otimes P_m$ where $P_m = K[x_{2n+1}, \ldots x_{2n+m}]$ is a polynomial algebra we use the following notation

$$x_1 := q_1, \ldots, x_n := q_n, x_{n+1} := p_1, \ldots, x_{2n} := p_n.$$

Then $A_n \otimes P_m = \bigoplus_{\alpha \in \mathbb{N}^s} K x^\alpha$ where $s := 2n + m$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the *order* of the x's is *fixed*. The algebra $A_n \otimes P_m$ admits the finite set of *commuting locally nilpotent* derivations, namely, the 'partial derivatives':

$$\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_s := \frac{\partial}{\partial x_s}.$$

Clearly, $\partial_i = \mathrm{ad}(x_{n+i})$ and $\partial_{n+i} = -\mathrm{ad}(x_i)$, $i = 1, \ldots, n$ (where $\mathrm{ad}(a) : A \to A$, $b \mapsto [a, b]$ is the *inner* derivation of the algebra A, $a \in A$).

For each $i = 1, \ldots, s$, consider the maps from Theorem 2.2,

$$\phi_i := \sum_{k \geq 0} (-1)^k \frac{x_i^k}{k!} \partial_i^k : A_n \otimes P_m \to A_n \otimes P_m.$$

For each $i = 2n+1, \ldots, s$, the map ϕ_i *commutes* with *all* the maps ϕ_j. For each $i = 1, \ldots, n$, the map ϕ_i commutes with all the maps ϕ_j but ϕ_{n+i}, and the map ϕ_{n+i} commutes with all the maps ϕ_j but ϕ_i. Note that $A_n \otimes P_m = K \oplus V$ where $V := \bigoplus_{0 \neq \alpha \in \mathbb{N}^s} K x^\alpha$. Using Theorem 2.2, we see that the map (the order is important)

$$\phi := \phi_s \phi_{s-1} \cdots \phi_1 : A_n \otimes P_m \to A_n \otimes P_m, \quad a = \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha x^\alpha \mapsto \phi(a) = \lambda_0, \qquad (1)$$

is a *projection* onto K.

The next result is a kind of a *Taylor* formula (note though that even for polynomials this is *not* the Taylor formula. The both formulae are essentially a formula for the *identity map* and they give a presentation of an element as a series but the formula below has one obvious advantage - it is 'more economical', i.e. there is no evaluation at $x = 0$ as in the Taylor formula).

Theorem 2.3 *For any $a \in A_n \otimes P_m$,*

$$a = \sum_{\alpha \in \mathbb{N}^s} \phi(\frac{\partial^\alpha}{\alpha!} a) x^\alpha$$

where $s = 2n + m$ and $\alpha! := \alpha_1! \cdots \alpha_s!$.

Proof. If $a = \sum \lambda_\alpha x^\alpha$, $\lambda_\alpha \in K$, then, by (1), $\phi(\frac{\partial^\alpha}{\alpha!}a) = \lambda_\alpha$. $\square$

So, the identity map id : $A_n \otimes P_m \to A_n \otimes P_m$ has a nice presentation

$$\mathrm{id}(\cdot) = \sum_{\alpha \in \mathbb{N}^s} \phi(\frac{\partial^\alpha}{\alpha!}(\cdot))x^\alpha. \tag{2}$$

Let $\mathrm{Aut}_K(A_n \otimes P_m)$ be the group of K-algebra automorphisms of the algebra $A_n \otimes P_m$. Given an automorphism $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. It is uniquely determined by the elements

$$x_1' := \sigma(x_1), \ldots, x_s' := \sigma(x_s) \tag{3}$$

of the algebra $A_n \otimes P_m$. The centre $Z := Z(A_n \otimes P_m)$ of the algebra $A_n \otimes P_m$ is equal to P_m. Therefore, the restriction $\sigma|_{P_m} \in \mathrm{Aut}_K(P_m)$, and so

$$\Delta := \det(\frac{\partial x_{2n+i}'}{\partial x_{2n+j}}) \in K^*$$

where $i, j = 1, \ldots, n$. The corresponding (to the elements $x_1', \ldots, x_s'$) 'partial derivatives' (the set of commuting locally nilpotent derivations of the algebra $A_n \otimes P_m$)

$$\partial_1' := \frac{\partial}{\partial x_1'}, \ldots, \partial_s' := \frac{\partial}{\partial x_s'} \tag{4}$$

are equal to

$$\partial_i' := \mathrm{ad}(\sigma(x_{n+i})), \quad \partial_{n+i}' := -\mathrm{ad}(\sigma(x_i)), \quad i = 1, \ldots, n, \tag{5}$$

$$\partial_{2n+j}' := \Delta^{-1} \det \begin{pmatrix} \frac{\partial \sigma(x_{2n+1})}{\partial x_{2n+1}} & \cdots & \frac{\partial \sigma(x_{2n+1})}{\partial x_{2n+m}} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_{2n+1}} & \cdots & \frac{\partial}{\partial x_{2n+m}} \\ \vdots & \vdots & \vdots \\ \frac{\partial \sigma(x_{2n+m})}{\partial x_{2n+1}} & \cdots & \frac{\partial \sigma(x_{2n+m})}{\partial x_{2n+m}} \end{pmatrix}, \quad j = 1, \ldots, m, \tag{6}$$

where we 'drop' $\sigma(x_{2n+j})$ in the determinant $\det(\frac{\partial \sigma(x_{2n+k})}{\partial x_{2n+l}})$.

For each $i = 1, \ldots, s$, let

$$\phi_i' := \sum_{k \geq 0}(-1)^k \frac{(x_i')^k}{k!}(\partial_i')^k : A_n \otimes P_m \to A_n \otimes P_m \tag{7}$$

and (the order is important)

$$\phi_\sigma := \phi_s' \phi_{s-1}' \cdots \phi_1'. \tag{8}$$

Theorem 2.4 (The Inversion Formula) *For each $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$ and $a \in A_n \otimes P_m$,*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)x^\alpha,$$

where $(\partial')^\alpha := (\partial_1')^{\alpha_1} \cdots (\partial_s')^{\alpha_s}$ and $s = 2n + m$.

Proof. By Theorem 2.3, $a = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)(x')^\alpha$. Applying σ^{-1} we have the result

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)\sigma^{-1}((x')^\alpha) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)x^\alpha. \quad \square$$

Corollary 2.5 *The question in the Abstract has an affirmative answer iff both the Jacobian conjecture and the Dixmier problem have (in more detail, $JD_n \Leftrightarrow JC_n + DP_n$).*

Proof. ($\Rightarrow$) Obvious (the JC and the DP are special cases).
($\Leftarrow$) Suppose that a K-algebra endomorphism $\sigma : A_n \otimes P_m \to A_n \otimes P_m$ satisfies the condition $\det(\frac{\partial \sigma(x_i)}{\partial x_j}) = 1$. Note that P_m is the *centre* of the algebra $A_n \otimes P_m$. Then the JC implies $\sigma|_{P_m} \in \mathrm{Aut}_K(P_m)$. Without loss of generality one can assume that $\sigma|_{P_m} = \mathrm{id}$. Let Q_m be the field of fractions of P_m. Then σ can be extended to an endomorphism of the algebra $A_n \otimes Q_m$. By the DP, $\sigma \in \mathrm{Aut}_K(A_n \otimes Q_m)$. By Theorem 2.4, $\sigma^{-1}(A_n \otimes P_m) \subseteq A_n \otimes P_m$, and so $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. $\square$
Remark. Note that an algebra endomorphism σ of the algebra $A_n \otimes P_m$ satisfying $\det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^*$ is automatically an algebra *monomorphism*: $\sigma|_{P_m}$ is an algebra monomorphism, it induces an algebra monomorphism, say σ, on the field of fractions Q_m of P_m, hence σ can be extended to an algebra endomorphism of the *simple* algebra $A_n \otimes Q_m$, hence σ is an algebra *monomorphism*.

3 The Degree of Inverse Automorphism

In this section, Theorem 3.3 and Corollary 3.4 are proved.
For an automorphisms σ and σ^{-1} we keep the notation from the previous sections.
An automorphism $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$ is uniquely determined by $\sigma(x_1), \ldots, \sigma(x_s)$. The *degree* of the automorphism σ is defined as

$$\deg \sigma := \max\{\deg \sigma(x_i) \,|\, i = 1, \ldots, s\}$$

where the *degree* $\deg a$ of an element $a = \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha x^\alpha \in A_n \otimes P_m$ is defined as

$$\deg a := \max\{|\alpha| := \alpha_1 + \cdots + \alpha_s \,|\, \lambda_\alpha \neq 0\}.$$

Theorem 3.1 *[3], [9]. Given $\sigma \in \mathrm{Aut}_K(P_m)$. Then $\deg \sigma^{-1} \leq (\deg \sigma)^{m-1}$.*

By Theorem 2.4, $\sigma^{-1}(x_i) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}x_i)x^\alpha$, then applying σ to this equality and using the fact that $\sigma(x^\alpha) = (x')^\alpha$, we have the equality

$$\sigma^{-1}(x_i') = \sigma^{-1}\sigma(x_i) = \sigma\sigma^{-1}(x_i) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}x_i)(x')^\alpha. \tag{9}$$

The next lemma follows directly from (9) and it gives the *exact* value for the degree of σ^{-1}.

Lemma 3.2 *Given $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. Then*

$$\deg \sigma^{-1} = \max\{\deg'(x_i) \,|\, i = 1, \ldots, s\}$$

where, for $a = \sum_{\alpha \in \mathbb{N}^s} \lambda'_\alpha (x')^\alpha \in A_n \otimes P_m$, $\deg'(a) := \max\{|\alpha| \,|\, \lambda'_\alpha \neq 0\}$.

Theorem 3.3 *Given $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. Then*

$$\deg \sigma^{-1} \leq (\deg \sigma)^{s-1}$$

where $s := 2n + m$.

Proof. The algebra $A := A_n \otimes P_m = \cup_{\alpha \in \mathbb{N}^s} K x^\alpha = \cup_{i \geq 0} \mathcal{A}_i$ is a *filtered* algebra ($\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for all $i, j \geq 0$) where $\mathcal{A}_i := \oplus_{|\alpha| \leq i} K x^\alpha$ and

$$\dim_K(\mathcal{A}_i) = \binom{i+s}{s} = \frac{(i+s)(i+s-1)\cdots(i+1)}{s!} = \frac{i^s}{s!} + \cdots, \quad i \geq 0,$$

where here and everywhere the three dots mean 'smaller' terms. The filtration $\{\mathcal{A}_i\}$ is a *standard* filtration (we use the terminology of the book of H. Krause and T. Lenagan, [7], where one can find all the missed definitions), so the *Gelfand-Kirillov* dimension of the algebra A is $\mathrm{GK}(A) = s$. The associative graded algebra $\mathrm{gr}\, A := \oplus_{i \geq 0} \mathcal{A}_i/\mathcal{A}_{i-1}$ is canonically isomorphic to a polynomial algebra in s variables. So, the algebra A is an *almost commutative* algebra. Given a finitely generated A-module $M = AM_0 = \cup_{i \geq 0} M_i$, $M_i := \mathcal{A}_i M_0$, where M_0 is a finite dimensional generating space for the module M, then there exists a polynomial (so-called, the *Hilbert polynomial* of M) $p_M \in \mathbb{Q}[t]$ such that

$$\dim_K(M_i) = p_M(i) = \frac{e(M)i^{\mathrm{GK}(M)}}{\mathrm{GK}(M)!} + \cdots, \quad i \gg 0,$$

where $e(M) \in \mathbb{N}$ is the *multiplicity* of M. *All* algebras involved in this proof will be *algebras of the type $A_k \otimes P_l$*, so we will use Hilbert polynomials and multiplicity for certain modules.

Fix $\nu \in \{1, \ldots, s\}$. Then

$$A = \oplus_{k \geq 0} \Lambda(x'_\nu)^k, \quad \text{where} \quad \Lambda = \Lambda(\nu) := K\langle x'_1, \ldots, \widehat{x'_\nu}, \ldots, x'_s \rangle \tag{10}$$

is an algebra of type $A_k \otimes P_l$ (the hat over the symbol means that it is missed), and $\mathrm{GK}(\Lambda) = s - 1$. A nonzero element $a \in A$ is a *unique* sum

$$a = a_0 + a_1 x'_\nu + \cdots + a_d (x'_\nu)^d, \quad a_i \in \Lambda, \quad a_d \neq 0.$$

Then d is called the x'_ν-*degree* of the element a denoted $\deg_{x'_\nu}(a)$.

Fix $j \in \{1, \ldots, s\}$. Let $d_j := \deg_{x'_\nu}(x_j)$. By Lemma 3.2, in order to finish the proof of this theorem it suffices to show that $d_j \leq (\deg \sigma)^{s-1}$ (the field K has characteristic zero, in particular it is *infinite*, so making suitable 'linear changes of variables' $\{x'_\mu\}$ (i.e. up to a *linear algebra automorphism* of $A_n \otimes P_m$) one can assume that $d_j = \deg \sigma^{-1}$).

Clearly,
$$A \supseteq M \oplus Ax_j, \quad M := \oplus_{k=0}^{d_j-1} \Lambda(x'_\nu)^k.$$

Then M can be seen as a Λ-*submodule* of A/Ax_j. Consider the filtration $\{\mathcal{P}_i\}$ on the A-module A/Ax_j induced by the 1-dimensional generating space $K\overline{1}$, $\overline{1} := 1 + Ax_j$. Then

$$\dim_K(\mathcal{P}_i) = \binom{i + s - 1}{s - 1} = \frac{i^{s-1}}{(s-1)!} + \cdots, \quad i \geq 0.$$

Let $\{\mathcal{P}'_i\}$ be the *standard* filtration of the algebra Λ (with respect to the generating set $x'_1, \ldots, \widehat{x'_\nu}, \ldots, x'_s$). Clearly,

$$\mathcal{P}'_i \subseteq \mathcal{P}_{i(\deg \sigma)}, \quad i \geq 0.$$

Fix a natural number, say t, such that $(x'_\nu)^k \in \mathcal{P}_t$ for all $k = 0, \ldots, d_j - 1$. Then, for all $i \gg 0$, $M_i := \oplus_{k=0}^{d_j-1} \mathcal{P}'_i(x'_\nu)^k \subseteq \mathcal{P}_{i(\deg \sigma)+t}$. Therefore,

$$\dim_K(\mathcal{P}_{i(\deg \sigma)+t}) = \frac{(\deg \sigma)^{s-1} i^{s-1}}{(s-1)!} + \cdots \geq \dim_K(M_i)$$

$$\geq \sum_{k=0}^{d_j-1} \binom{i + s - 1 - t}{s - 1} = \frac{d_j i^{s-1}}{(s-1)!} + \cdots, \quad i \gg 0.$$

Hence, $d_j \leq (\deg \sigma)^{s-1}$, as required. $\square$

Recall that a *commutative* ring R is called *reduced* iff its *nil-radical* is equal to zero $(\mathfrak{n}(R) := \cap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} = 0)$ iff a zero is the only nilpotent element of R.

Corollary 3.4 *If a commutative ring (not necessarily a field) K is a reduced $\mathbb{Q}$-algebra and $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. Then*

$$\deg \sigma^{-1} \leq (\deg \sigma)^{s-1}.$$

Proof. We write $A_n(K)$ and $P_m(K)$ to indicate the base ring K. If K is a *domain* with a field of fractions F then σ can be extended to an element of $\mathrm{Aut}_F(A_n(F) \otimes_F P_m(F))$ and the result follows from Theorem 3.3.

In the general situation, by the previous case, for each prime ideal $\mathfrak{p}$ of K, reduction modulo $\mathfrak{p}$ gives an element $\sigma_\mathfrak{p} \in \mathrm{Aut}_{K/\mathfrak{p}}(A_n(K/\mathfrak{p}) \otimes_{K/\mathfrak{p}} P_m(K/\mathfrak{p}))$. Since $K/\mathfrak{p}$ is a domain,

$$\deg \sigma_\mathfrak{p}^{-1} \leq (\deg \sigma_\mathfrak{p})^{s-1} \leq (\deg \sigma)^{s-1} \text{ for all } \mathfrak{p} \in \mathrm{Spec}(K),$$

which implies $\deg \sigma^{-1} \leq (\deg \sigma)^{s-1}$ since K is reduced. $\square$

Remark. In the polynomial case, P_m, if K is *not reduced* then there is no uniform upper bound for the degree $\deg \sigma^{-1}$ depending only on m and $\deg \sigma$, see p.56–57, [10]: let $P_1 = K[x]$ and $K := \mathbb{Q}[T]/(T^l)$ where $\sigma : x \mapsto x' := x - tx^2$ where $t := T + (T^l) \in K$. Then $\deg_x(x') \geq \frac{l-1}{2} + 1$ (p.57, [10]). *The same is true for automorphisms of the algebras $A_n \otimes P_m$ since the automorphism σ can extended to a K-automorphism of the 1st Weyl algebra $A_1(K)$ by the rule*

$$\frac{d}{dx} \mapsto \frac{d}{dx'} = \frac{dx}{dx'}\frac{d}{dx} = \left(\frac{dx'}{dx}\right)^{-1}\frac{d}{dx} = \frac{1}{1-2tx}\frac{d}{dx} = (1 + 2tx + \cdots + (2tx)^{m-1})\frac{d}{dx}.$$

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4 The inversion formula for automorphism of a power series algebra

In this section, the inversion formula for an automorphism of the power series algebra is obtained (Theorem 4.3).

Lemma 4.1 *Let A be an algebra over a field K of characteristic zero, δ be a K-derivation of the algebra A (not necessarily locally nilpotent) such that $\delta(x) = 1$ for a central element $x \in A$. Suppose that the algebra A is complete in $\mathfrak{m}$-adic topology and $x \in \mathfrak{m}$ ($\mathfrak{m}$ is a right ideal of A). Then the K-linear map $\phi := \sum_{i \geq 0} (-1)^i \frac{x^i}{i!} \delta^i : A \to A$ satisfies the following properties:*

1. $\phi(x) = 0$.

2. ϕ is an algebra homomorphism of A.

3. ϕ is a homomorphism of left/right A^δ-modules.

4. $\operatorname{im}(\phi) = A^\delta$ and $\phi(y) = y$ for all $y \in A^\delta$.

Proof. The map ϕ is well-defined. Then the proof is a repetition of the proof of Theorem 2.2. $\square$

Remark. If for an arbitrary K-algebra A the infinite sum $\phi := \sum_{i \geq 0} (-1)^i \frac{x^i}{i!} \delta^i$ makes sense then Lemma 4.1 holds.

Let $\widehat{P}_n := K[[x_1, \ldots, x_m]]$ be *an algebra of formal power series* in x_i (= the completion of the polynomial algebra P_m at the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_m)$). The partial derivatives $\partial_1, \ldots, \partial_m$ are a set of commuting continuous (in $\mathfrak{m}$-adic topology) derivations of the algebra $\widehat{P}_m$ satisfying $\partial_i(x_j) = \delta_{ij}$.

Using Lemma 4.1, we have a set of *commuting* algebra endomorphisms:

$$\phi_i := \sum_{k \geq 0} (-1)^k \frac{x_i^k}{k!} \partial_i^k : \widehat{P}_m \to \widehat{P}_m, \quad i = 1, \ldots, m.$$

Each ϕ_i is a *projection* onto $\operatorname{im} \phi_i = K[[x_1, \ldots, \widehat{x}_i, \ldots, x_n]]$, $\phi_i(x_i) = 0$, and $\phi_i : \widehat{P}_m \to \widehat{P}_m$ is a homomorphism of $K[[x_1, \ldots, \widehat{x}_i, \ldots, x_n]]$-modules ($\widehat{x}_i$ means that x_i is missed).

The algebra endomorphism

$$\phi := \phi_1 \cdots \phi_m : \widehat{P}_m = K \oplus \widehat{P}_m \mathfrak{m} \to \widehat{P}_m = K \oplus \widehat{P}_m \mathfrak{m}, \quad a = \sum_{\alpha \in \mathbb{N}^m} \lambda_\alpha x^\alpha \mapsto \phi(a) = \lambda_0, \quad (11)$$

is a *projection* onto K.

Theorem 4.2 *For any $a \in \widehat{P}_m$,*

$$a = \sum_{\alpha \in \mathbb{N}^m} \phi(\frac{\partial^\alpha}{\alpha!} a) x^\alpha.$$

Proof. If $a = \sum \lambda_\alpha x^\alpha$, $\lambda_\alpha \in K$, then, by (11), $\phi(\frac{\partial^\alpha}{\alpha!}a) = \lambda_\alpha$. $\square$

So, the identity map id : $\widehat{P}_m \to \widehat{P}_m$ has a nice presentation

$$\mathrm{id}(\cdot) = \sum_{\alpha \in \mathbb{N}^m} \phi(\frac{\partial^\alpha}{\alpha!}(\cdot))x^\alpha. \tag{12}$$

Let $\mathrm{Aut}_K(\widehat{P}_m)$ be the group of continuous K-algebra automorphisms of the algebra $\widehat{P}_m$. Given an automorphism $\sigma \in \mathrm{Aut}_K(\widehat{P}_m)$. It is uniquely determined by the elements

$$x'_1 := \sigma(x_1), \ldots, x'_m := \sigma(x_m) \tag{13}$$

of the algebra $\widehat{P}_m$ (all the $x'_i \in \widehat{P}_m\mathfrak{m}$). Then $\Delta := \det(\frac{\partial x'_i}{\partial x_j}) \in (\widehat{P}_m)^*$, the group of all invertible elements of $\widehat{P}_m$. The corresponding (to the elements $x'_1, \ldots, x'_m$) 'partial derivatives' (the set of commuting continuous derivations of the algebra $\widehat{P}_m$)

$$\partial'_1 := \frac{\partial}{\partial x'_1}, \ldots, \partial'_m := \frac{\partial}{\partial x'_m} \tag{14}$$

are equal to

$$\partial'_i := \Delta^{-1}\det \begin{pmatrix} \frac{\partial\sigma(x_1)}{\partial x_1} & \cdots & \frac{\partial\sigma(x_1)}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial\sigma(x_m)}{\partial x_1} & \cdots & \frac{\partial\sigma(x_m)}{\partial x_m} \end{pmatrix}, \quad j = 1, \ldots, m, \tag{15}$$

where we 'drop' $\sigma(x_i)$ in the determinant $\det(\frac{\partial\sigma(x_i)}{\partial x_j})$.

For each $i = 1, \ldots, m$, let

$$\phi'_i := \sum_{k \geq 0}(-1)^k\frac{(x'_i)^k}{k!}(\partial'_i)^k : \widehat{P}_m \to \widehat{P}_m \tag{16}$$

and

$$\phi_\sigma := \phi'_1 \cdots \phi'_m. \tag{17}$$

Theorem 4.3 (The Inversion Formula) *For each $\sigma \in \mathrm{Aut}_K(\widehat{P}_m)$ and $a \in \widehat{P}_m$,*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^m} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)x^\alpha.$$

Proof. By Theorem 4.2, $a = \sum_{\alpha \in \mathbb{N}^m} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)(x')^\alpha$. Applying σ^{-1} we have the result

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^m} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)\sigma^{-1}((x')^\alpha) = \sum_{\alpha \in \mathbb{N}^m} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!}a)x^\alpha. \quad \square$$

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Corollary 4.4 *Let σ be an algebra endomorphism of the polynomial algebra $P_m = K[x_1, \ldots, x_m]$ satisfying $\det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^*$ and $\sigma(\mathfrak{m}) \subseteq \mathfrak{m}$ where $\mathfrak{m} := (x_1, \ldots, x_n)$. Then*

1. *the algebra endomorphism $\phi_\sigma : P_m = K \oplus \mathfrak{m} \to P_m = K \oplus \mathfrak{m}$, $\lambda + \sum x_i a_i \mapsto \lambda$, $\lambda \in K$, $a_i \in P_m$ (see (17)) is a projection onto K.*

2. *$\cap_{i=1}^m \ker_{P_m}(\partial_i') = \cap_{i=1}^m \ker_{\widehat{P}_m}(\partial_i') = K$.*

3. *$\cap_{i=1}^m N(\partial_i', P_m) = \cap_{i=1}^m N(\partial_i', \widehat{P}_m) = \sigma(P_m)$.*

Proof. 1. The two conditions guarantee that the extension of the σ to a continuous (in $\mathfrak{m}$-adic topology) algebra endomorphism, say σ, of $\widehat{P}_m$ is, in fact, an automorphism. By (11), the endomorphism $\phi_\sigma : \widehat{P}_m \to \widehat{P}_m$ is the projection onto K, hence its restriction $\phi_\sigma : P_m \to P_m$ is a projection onto K as well.

2. $K \subseteq \cap_{i=1}^m \ker_{P_m}(\partial_i') \subseteq \cap_{i=1}^m \ker_{\widehat{P}_m}(\partial_i') = \phi_\sigma(\widehat{P}_m) = K$.

3. Note that $\partial_1', \ldots, \partial_m'$ is a set of commuting *locally nilpotent* derivations of the algebra $\cap_{i=1}^m N(\partial_i', P_m)$. By Corollary 5.6, $\cap_{i=1}^m N(\partial_i', P_m) = C[\sigma(x_1), \ldots, \sigma(x_m)]$ and, similarly, $\cap_{i=1}^m N(\partial_i', \widehat{P}_m) = C[\sigma(x_1), \ldots, \sigma(x_m)]$ where $C = \cap_{i=1}^m \ker_{P_m}(\partial_i') = \cap_{i=1}^m \ker_{\widehat{P}_m}(\partial_i') = K$, by statement 2. $\square$

5 A Characterization of the Weyl Algebras and the Polynomial Algebras in terms of commuting locally nilpotent derivations

The Weyl algebras and the polynomial algebras are, in some sense, special algebras in the class of all algebras. Theorem 5.3 explains this fact in terms of commuting locally nilpotent derivations.

We say that derivations $\delta_1, \ldots, \delta_s$ of an algebra A have *generic kernels* iff the intersections $\cap_{i \in I} \ker \delta_i$, $I \subseteq \{1, \ldots, s\}$, are *distinct* (iff $\cap_{j \neq i} \ker \delta_j$, $i = 1, \ldots, s$ are distinct).

Lemma 5.1 *Let A be an algebra over a field K of characteristic zero, and $\delta_1, \ldots, \delta_s$ be commuting locally nilpotent K-derivations of A that have generic kernels and with $\Gamma := \cap_{i=1}^s \ker \delta_i$, a division ring such that $\Gamma \neq A$. Then there exist nonzero elements $x_i \in C_i' := \cap_{j \neq i} \ker \delta_j$, $i = 1, \ldots, s$ such that $\delta_i(x_j) = \delta_{ij}$, the Kronecker delta.*

Proof. Let us consider first the case when $s = 1$. The derivation δ_1 is locally nilpotent and nonzero since $\ker \delta = \Gamma \neq A$. So we can find an element $y \in A$ such that $0 \neq \lambda := \delta_1(y) \in \ker \delta_1 = \Gamma$. Then $\delta_1(x) = 1$ for $x = \lambda^{-1} y$.

Suppose now that $s > 1$. For each $i = 1, \ldots, s$, let $\overline{\delta_i}$ be the restriction of the derivation δ_i to the subalgebra C_i' of A. The kernel of $\overline{\delta_i}$ is equal to Γ, so by the previous argument one can find an element, say $x_i \in C_i'$, satisfying $\delta_i(x_i) = 1$. Obviously, $\delta_i(x_j) = \delta_{ij}$. $\square$

Let us recall one of the key results of symplectic algebra.

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Lemma 5.2 *Let Φ be an antisymmetric bilinear form on a finite dimensional vector space V, and let $y_1, \ldots, y_m$ be a basis of the kernel of Φ. Then we can complete the set $y_1, \ldots, y_m$ to the basis $p_1, \ldots, p_n, q_1, \ldots, q_n, y_1, \ldots, y_l$ of V such that $\Phi(p_i, q_j) = \delta_{ij}$, $\Phi(p_i, p_j) = \Phi(q_i, q_j) = 0$ for all i, j.*

The theorem below gives a characterization of the algebras of the type $A_n \otimes P_m$ (the Weyl algebras with polynomial coefficients) in terms of commuting locally nilpotent derivations.

Theorem 5.3 *Let A be an algebra over a field K of characteristic zero. Then the following statements are equivalent.*

1. *There exist commuting locally nilpotent nonzero K-derivations $\delta_1, \ldots, \delta_s$ of the algebra A with generic kernels $C_i = \ker \delta_i$ satisfying $\cap_{i=1}^s C_i = K$.*

2. *The algebra A is an iterated Ore extension*

$$A = K[x_1][x_2; d_2][x_3; d_3] \cdots [x_s; d_s]$$

   such that $\lambda_{ij} := d_i(x_j) \in K$ for all $i > j$, and $\delta_i(x_j) = \delta_{ij}$, the Kronecker delta.

3. *The algebra A is isomorphic to the tensor product $A_n \otimes P_m$ (over K) of the Weyl algebra A_n with a polynomial algebra P_m in m indeterminates, and $2n + m = s$.*

   Suppose that the (equivalent) conditions above hold. Then

   (a) *the elements $x_1, \ldots, x_s$ are uniquely determined up to scalar addition.*

   (b) *$n = \frac{1}{2}\mathrm{rk}(\Lambda)$ and $m = s - 2n = \dim \ker(\Lambda)$ where $\Lambda = (\lambda_{ij})$ is the antisymmetric $s \times s$ matrix with lower diagonal entries λ_{ij} as above.*

   (c) *For each i, the algebra C_i is an iterated Ore extension*

$$K[x_1][x_2; d_2] \cdots [x_{i-1}; d_{i-1}][x_{i+1}; d_{i+1}] \cdots [x_s; d_s]$$

   with $d_i(x_j) = \lambda_{ij}$ as above. Hence, $C_i \simeq A_{m_i} \otimes P_{l_i}$ with $2m_i + l_i = s - 1$.

   (d) *The Gelfand-Kirillov dimension $\mathrm{GK}(A) = s$.*

Remark. As an abstract algebra the iterated Ore extension A from the second statement is generated by the elements $x_1, \ldots x_s$ subject to the defining relations

$$x_i x_j - x_j x_i = \lambda_{ij}, \quad \text{for all} \;\; i > j.$$

So, for any permutation $i_1, \ldots, i_s$, of the indices $1, \ldots, s$, the algebra A is the iterated Ore extension

$$K[x_{i_1}][x_{i_2}; d_{i_2}] \cdots [x_{i_s}; d_{i_s}]$$

with $d_\alpha(x_\beta) = \lambda_{\alpha\beta}$, if $\alpha > \beta$, and $-\lambda_{\alpha\beta}$, if $\alpha < \beta$.

Proof. $(1 \Rightarrow 2)$ We use induction on s. Let $s = 1$. Since δ_1 is a locally nilpotent nonzero derivation with $\ker \delta_1 = K$ we can find an element $x \in A$ such that $\delta_1(x) = 1$. By Lemma 2.1, $A = K[x]$ is a polynomial algebra.

Suppose that $s > 1$ and the result is true for $s-1$. By Lemma 5.1, we can find a nonzero element $x_s \in A$ such that $\delta_i(x_s) = \delta_{is}$. By Lemma 2.1, $A = N(\delta_s, A) = C_s[x_s; d_s]$ is an Ore extension with coefficients from C_s where the derivation d_s of C_s is the restriction of the inner derivation $\mathrm{ad}\, x_s$ of A to C_s. The derivations δ_i commute, thus $\delta_i(C_s) \subseteq C_s$ for all i. We denote by $\overline{\delta_1}, \ldots, \overline{\delta_{s-1}}$ the restrictions of the derivations $\delta_1, \ldots, \delta_{s-1}$ to the subalgebra C_s of A. Then $\overline{\delta_1}, \ldots, \overline{\delta_{s-1}}$ are commuting locally nilpotent nonzero K-derivations of the algebra C_s with generic kernels $\overline{C_1}, \ldots, \overline{C_{s-1}}$ satisfying $\cap_{i=1}^{s-1} \overline{C_i} = \cap_{i=1}^{s} C_i = K$. By induction, the algebra $\overline{C_s}$ is an iterated Ore extension

$$K[x_1][x_2; d_2] \cdots [x_{s-1}; d_{s-1}]$$

with $d_i(x_j) = \lambda_{ij} \in K$ for all i, j less than s. It remains to prove that the elements $\lambda_{si} := d_s(x_i) = [x_s, x_i] \in C_s$ belong to K, for all $i < s$. For $i, j < s$,

$$\delta_j(\lambda_{si}) = \delta_j([x_s, x_i]) = [\delta_j(x_s), x_i] + [x_s, \delta_j(x_i)] = [0, x_i] + [x_s, \lambda_{ji}] = 0,$$

thus $\lambda_{si} \in \cap_{k=1}^{s} C_k = K$, as required. Let $x_1', \ldots, x_s'$ be elements of A satisfying $\delta_i(x_j') = \delta_{ij}$. Then $\delta_i(x_j - x_j') = 0$, so $x_j - x_j' \in \cap_{k=1}^{s} C_k = K$, this proves (a).

$(2 \Rightarrow 3)$ The s-dimensional vector subspace $V = Kx_1 \oplus \cdots \oplus Kx_s$ of the iterated Ore extension A as in statement 2 is equipped with the antisymmetric bilinear form:

$$V \times V \to K, \quad (u, v) \to [u, v] := uv - vu.$$

Then $\Lambda = (\lambda_{ij})$ is the matrix of this form in the basis $x_1, \ldots, x_s$ of V. By Lemma 5.2, we can choose a basis $p_1, \ldots, p_n, q_1, \ldots, q_n, y_1, \ldots, y_m$ of the vector space V such that

$$[p_i, q_j] = \delta_{ij}, \quad [p_i, p_j] = [q_i, q_j] = 0,$$

$$[p_i, y_k] = [q_i, y_k] = [y_k, y_{k'}] = 0,$$

for all possible i, j, k, and k'. So, $A = A_n \otimes P_m$ where $A_n = K[p_1, \ldots, p_n, q_1, \ldots, q_n]$ is the Weyl algebra and $P_m = K[y_1, \ldots, y_m]$ is a polynomial algebra in m indeterminates (see the Remark above).

Clearly, $2n + m = s$, $n = \frac{1}{2}\mathrm{rk}(\Lambda)$, and $m = s - 2n = \dim \ker(\Lambda)$, this proves (b).

$(3 \Rightarrow 1)$ If $A = A_n \otimes P_m$ (as above) then

$$\mathrm{ad}\, p_1, \ldots, \mathrm{ad}\, p_n, \, \mathrm{ad}\, q_1, \ldots, \mathrm{ad}\, q_n, \, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}$$

are commuting locally nilpotent nonzero K-derivations of the algebra A with generic kernels (recall that, for each element $a \in A$, $\mathrm{ad}\, a : A \to A$, $x \mapsto [a, x] := ax - xa$, is the *inner derivation* of the algebra A)

$$\ker \mathrm{ad}\, p_i = K[p_i] \otimes K[p_1, \ldots, \widehat{p_i}, \ldots, p_n, q_1, \ldots, \widehat{q_i}, \ldots, q_n] \otimes P_m \simeq A_{n-1} \otimes P_{m+1},$$

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$$\ker \operatorname{ad} q_i = K[q_i] \otimes K[p_1, \ldots, \widehat{p}_i, \ldots, p_n, q_1, \ldots, \widehat{q}_i, \ldots, q_n] \otimes P_m \simeq A_{n-1} \otimes P_{m+1},$$

$$\ker \frac{\partial}{\partial y_j} = A_n \otimes K[y_1, \ldots, \widehat{y}_j, \ldots, y_m] \simeq A_n \otimes P_{m-1},$$

such that their intersection is K (where hat over a symbol means that it is missed in the list). The statement (c) is evident now. Finally, the Gelfand-Kirillov dimension of the algebra A over the field K is equal to

$$\operatorname{GK}(A) = \operatorname{GK}(A_n \otimes P_m) = 2n + m = s,$$

this proves (iv). $\square$

A K-algebra A is called a *central* algebra iff its centre is K.

Corollary 5.4   *1. The Weyl algebras are the only central K-algebras that admit a finite set of commuting locally nilpotent derivations with generic kernels that have trivial intersection $(= K)$.*

   2. The polynomial algebras are the only commutative K-algebras that admit a finite set of commuting locally nilpotent derivations with generic kernels that have trivial intersection $(= K)$.

Corollary 5.5 *Let $\delta_1, \ldots, \delta_s$ be commuting locally nilpotent nonzero K-derivations of an K-algebra A satisfying $\cap_{i=1}^s \ker \delta_i = K$. If $s \neq \operatorname{GK}(A)$ then the kernels of the derivations $\delta_1, \ldots, \delta_n$ are not generic.*

   Proof. Suppose that the kernels of the derivations are generic. Then by Theorem 5.3.(iv), $s = \operatorname{GK}(A)$, a contradiction. $\square$

Corollary 5.6 *Let $\delta_1, \ldots, \delta_s$ be a set of commuting locally nilpotent derivations of an algebra A over a field K of characteristic zero such that $\delta_i(x_j) = \delta_{ij}$ for some elements $x_1, \ldots, x_s$ of A. Then $A = C[x_1][x_2; d_2] \cdots [x_s; d_s]$ where $C = \cap_{i=1}^s \ker \delta_i$ and $d_i(x_j) := [x_i, x_j] \in C$.*

   Proof. A repeated application of Lemma 2.1. $\square$

   The following important case of this corollary was proved in [4] (and when A is *commutative* the result was known before [folklore]).

Corollary 5.7 *Let $\delta_1, \ldots, \delta_s$ be a set of commuting locally nilpotent derivations of an algebra A over a field K of characteristic zero such that $\delta_i(x_j) = \delta_{ij}$ for central elements $x_1, \ldots, x_s$ of A. Then $A = C[x_1, \ldots, x_s]$ is a polynomial algebra with coefficients from $C = \cap_{i=1}^s \ker \delta_i$.*

Theorem 5.8 *Let σ be a K-algebra endomorphism of the algebra $A := A_n \otimes P_m$ such that $\det(\frac{\partial \sigma(x_{2n+i})}{\partial x_{2n+j}}) \in K^*$. Then σ is an algebra automorphism iff the derivations $\partial_1', \ldots, \partial_s'$ (see (5) and (6)) are locally nilpotent (iff $(\partial_i')^{(\deg \sigma)^{s-1}+1}(x_j) = 0$ for all i, j, by Theorem 2.3).*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Since $\partial_i'(x_j') = \delta_{ij}$ (see (3)) for all $i, j = 1, \ldots, s$, the derivations $\partial_1', \ldots, \partial_s'$ have generic kernels, then by Corollary 5.6, $A = \oplus_{\alpha \in \mathbb{N}^s} C(x')^\alpha$ where $C := \cap_{i=1}^s \ker(\partial_i')$. It follows from

$$s = \mathrm{GK}(A) \geq \mathrm{GK}(C) + s$$

that $\mathrm{GK}(C) = 0$, which means that every element of the algebra C is algebraic. Since scalars are the *only* algebraic elements in A we must have $C = K$ which means that $A = \sigma(A)$, i.e. σ is an automorphism. $\square$

Proposition 5.9 *Let σ be an algebra endomorphism of $A := A_n \otimes P_m$ that satisfies* $\det(\frac{\sigma(x_{2n+i})}{x_{2n+j}}) \in K^*$. *Then* $N(\partial_1', \ldots, \partial_s'; A) := \cap_{i=1}^s N(\partial_i', A) = \sigma(A)$ *and* $\cap_{i=1}^s \ker_A(\partial_i') = K$.

Proof. By Corollary 5.6, the intersection is equal to $N := C[x_1'][x_2'; d_2'] \cdots [x_s'; d_s']$ where $C := \cap_{i=1}^s \ker_A(\partial_i')$, $d_i' := (\mathrm{ad}\, x_i')|_C$, and $d_i'(x_j') := [x_i', x_j'] \in \{0, 1\}$. It follows from

$$s = \mathrm{GK}(A) \geq \mathrm{GK}(N) \geq \mathrm{GK}(C) + s$$

that $\mathrm{GK}(C) = 0$, i.e. each element of C is *algebraic* over K, hence $C = K$ (as the only algebraic elements of A are scalars). $\square$

6 Left and right face differential operators

The algebra $A := A_n \otimes P_m$ is *self-dual*, i.e. it is isomorphic to its *opposite* algebra A^{op},

$$A \to A^{op}, \quad x_i \mapsto x_{i+n}, \quad x_{n+i} \mapsto x_i, \quad x_{2n+j} \mapsto x_{2n+j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.$$

Theorem 6.1 *Given $\sigma, \tau \in \mathrm{Aut}_K(A)$ where $A := A_n \otimes P_m$ and $s := 2n + m$. Then*

1. *$\sigma = \tau$ iff $r_i \sigma = r_i \tau$, $i = 1, \ldots, s$, where $r_i : A \to A/x_i A$, $a \mapsto a + x_i A$.*

2. *$\sigma = \tau$ iff $l_i \sigma = l_i \tau$, $i = 1, \ldots, s$, where $l_i : A \to A/A x_i$, $a \mapsto a + A x_i$.*

Proof. The algebra A is self-dual, so it suffices to prove that $r_i \sigma = r_i \tau$, $i = 1, \ldots, s$ implies $\sigma = \tau$. We have to show that, for each $k = 1, \ldots, s$, $\sigma^{-1}(x_k) = \tau^{-1}(x_k)$ (since then $\sigma^{-1} = \tau^{-1}$ implies $\sigma = \tau$). $0 = r_k(x_k) = r_k \sigma \sigma^{-1}(x_k) = r_k \tau \sigma^{-1}(x_k)$, hence $\tau \sigma^{-1}(x_k) \in x_k A$, and so $\tau \sigma^{-1}(x_k) = x_k a_k$ for some $a_k \in A$. Applying τ^{-1} we obtain $\sigma^{-1}(x_k) = \tau^{-1}(x_k) b_k$ where $b_k = \tau^{-1}(a_k)$. By symmetry, $\tau^{-1}(x_k) = \sigma^{-1}(x_k) c_k$ for some $c_k \in A$. Now,

$$\sigma^{-1}(x_k) = \tau^{-1}(x_k) b_k = \sigma^{-1}(x_k) c_k b_k \quad \text{and} \quad \tau^{-1}(x_k) = \sigma^{-1}(x_k) c_k = \tau^{-1}(x_k) b_k c_k,$$

hence $c_k b_k = b_k c_k = 1$ since A is a domain. The only invertible elements of the algebra A are nonzero scalars, so $c_k, b_k \in K^*$. Note that the $A/x_i A$ is canonically identified with the algebra $K\langle x_1, \ldots, \widehat{x_i}, \ldots, x_s \rangle$ of type $A_n \otimes P_{m-1}$ if x_i is central or otherwise with $A_{n-1} \otimes P_{m+1}$. Fix $l \neq k$.

$$x_k = r_l(x_k) = r_l \sigma \sigma^{-1}(x_k) = r_l \tau \tau^{-1}(x_k) b_k = b_k x_k,$$

hence $b_k = 1$. Therefore, $\sigma^{-1}(x_k) = \tau^{-1}(x_k)$ for all k, as required. $\square$

In the case of the polynomial algebra $A = P_m$, the maps $r_i = l_i : P_m \to P_m/(x_i)$ are *algebra homomorphisms*, so any automorphism $\sigma \in \mathrm{Aut}_K(P_m)$ is uniquely determined by the algebra epimorphisms $r_i\sigma : P_m \to P_m/(x_i)$ *which in turn are uniquely determined by the face polynomials* $\{r_i\sigma(x_j)\} \,|\, i,j = 1,\ldots,m\}$ of σ (this is the result of J. H. McKay and S. S.-S. Wang, [8]).

In general situation, $A = A_n \otimes P_m$, $n \geq 1$, for each $i = 1,\ldots,n$, the maps r_i (resp. l_i), are *not* algebra homomorphisms, they are homomorphisms of *right* (resp. *left*) A-modules. Note that A is a simple algebra. Note that $r_{2n+j} = l_{2n+j}$ is an *algebra homomorphism* since the element x_{2n+j} is *central* for $j = 1,\ldots,m$.

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